1. $\frac{1}{x^3+x^2+x}$.

$$= \frac{1}{x(x^2+x+1)}.$$

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

writing as a single fraction.

$$= \frac{x(Bx+C)+(x^2+x+1)A}{x(x^2+x+1)}$$

$$1 = x(Bx+C)+(x^2+x+1)A.$$

$$1 = x^2A + x^2B + xA + xC + A.$$

collect up like terms

$$1 = x^2(A+B) + x(A+C) + A.$$

$$A+B=0.$$

$$A+C=0.$$

$$A=1$$

$$A=1, \quad B=-1, \quad C=-1$$

$$= \frac{1}{x} + \frac{-x-1}{x^2+x+1}$$

2. $\sqrt[3]{9y^2} = x^2+2.$

$$(9y^2)^{1/3} = x^2+2.$$

$$9y^2 = (x^2+2)^3.$$

$$y^2 = \frac{1}{9}(x^2+2)^3.$$

$$V^2 = \frac{1}{9}(x^2+2)^3.$$

$$V = \frac{1}{3}(x^2+2)^{3/2}. \qquad 0 \le x \le \sqrt{2}.$$

revolution about y-axis.

$$A = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{3}{2}\right)(x^2+2)^{1/2} \quad (2x)$$

$$= x(x^2+2)^{1/2}.$$

$$A = 2\pi \int_0^{\sqrt{2}} x\sqrt{1+x^2(x^2+2)}\, dx.$$

$$= 2\pi \int_0^{\sqrt{2}} x\sqrt{1+x^4+2x^2}\, dx.$$

$$= 2\pi \int_0^{\sqrt{2}} x\sqrt{(1+x^2)^2}\, dx.$$

$$= 2\pi \int_0^{\sqrt{2}} x(1+x^2)\, dx = 2\pi \int_0^{\sqrt{2}} (x+x^3)\, dx.$$

$$= 2\pi\left[\frac{x^2}{2}+\frac{x^4}{4}\right]_0^{\sqrt{2}}$$

$$= 2\pi\left[1+\frac{4}{4}-0\right]$$

$$= 2\pi[2]$$

$$= 4\pi \text{ units}^2$$

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3 The given family is $y = cx^2$.

differentiating both sides of the equation with respect to x.

$$\frac{dy}{dx} = 2cx \cdots\cdots (i)$$

eliminating c; we get $2y = x\frac{dy}{dx}$ ... (ii)

To get the differential equation of the family of orthogonal trajectories we replace $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy}\right)$

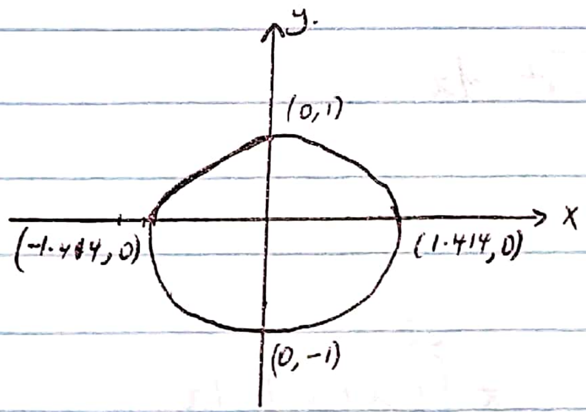
we get $2y = -x\frac{dx}{dy}$.

Intergrating we get the family of orthogonal trajectories as:

$$y^2 = -\frac{x^2}{2} + c \Rightarrow x^2 + 2y^2 = 2c \quad [c \text{ is a constant}]$$

Now sketch the family of orthogonal trajectories as;

if $c = 1$



Note $c > 0$ in order to sketch the family of orthogonal trajectories.

4. $xy' - 2y = 3x^4$

$\Rightarrow \frac{dy}{dx} + \left(\frac{-2}{x}\right) y = 3x^3$

$\frac{dy}{dx} + P(x)y = Q(x).$

$I.f = e^{\int -\frac{2}{x} dx.}$

$= e^{-2\ln x.}$

$= x^{-2.}$

$= 1/x^2.$

Solution.

$y.If = \int I.f\, Q(x)\, dx.$

$y.\frac{1}{x^2} = \int 3x^3 \times \frac{1}{x^2}\, dx$

$= \frac{y}{x^2} = \int 3x\, dx.$

$\Rightarrow y = \frac{3x^2}{2}.x^2 + cx^2$

$\Rightarrow y = \frac{3x^4}{2} + cx^2$ ---(i).

$y(1) = 2.$

From (i) $\Rightarrow$

$2 = \frac{3.1^4}{2} + C.1$

$2 = 3/2 + c.$

$c = 2 - 3/2.$

$c = 1/2.$

$\therefore y = \frac{3x^4}{2} + \frac{x^2}{2}.$

$\Rightarrow y = \frac{3x^4 + x^2}{2.}$

To verify lets check $y(1)$.

$y = \frac{3.1+1}{2} = \frac{4}{2} = 2.$

$\therefore y(1) = 2$ which is also given in the problem

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5. given that sequence $= \{n \cdot e^{-n}\}_{n=1}^{\infty}$

To prove: Sequence is decreasing and is bounded above by e^{-1}

Proof: Consider the general form of the sequence be

$\Rightarrow f(n) = a_n = ne^{-n}$ as a function.

Now

$$\Rightarrow f'(n) = \frac{d}{dn} f(n) = n \cdot \frac{d}{dn}(e^{-n}) + e^{-n} \frac{dn}{dn}.$$

$$f'(n) = -n \cdot e^{-n} + e^{-n}$$

$$f'(n) = e^{-n}(1-n)$$

Clearly as $n \geq 1$ and e^{-n} is always > 0

$1-n \leq 0$ and $e^{-n} > 0$ then

$\Rightarrow e^{-n}(1-n) \leq 0 \quad \forall n \in N$

$\Rightarrow f'(n) \leq 0$

Hence from calculus if $f'(n) \leq 0$; then it is decreasing

so, the sequence is decreasing.

Now as the sequence is decreasing; then

$a_1 \leq a_n \quad \forall n \in N$

and $a_1 = 1 \cdot e^{-1} = e^{-1}$

$\Rightarrow a_n \leq a_1$

$\Rightarrow [\, a_n \leq e^{-1} \quad \forall n \in N \,]$

and we know that if there exist a real number K such that

$a_n \leq K \; \forall n \in N$; then it is bounded above by K

Hence, sequence (a_n) is bounded above by e^{-1}

6. $\{a_n\}_{n=1}^{\infty}$ is a given sequence with $a_1=1$ & $a_{n+1}=2a_n+1$
we have to prove that $a_n = 2^n-1$ by Induction.

for $n=1$, $a_1 = 1$ (given)
$= 2^1-1$
ie. given relation is true for $n=1$

for $n=2$, $a_2 = a_1+1$
$= 2a_1+1$
$= 2\cdot1+1$
$= 2+1 = 3.$
$= 4-1$
$= 2^2-1$
given relation is true for $n=2$.
Let the given relation is true for $n=k$
ie, $a_k = 2^k-1$ ---- (i)
Now for $n=k+1$
$a_{k+1} = 2a_k+1$
$= 2(2^k-1)+1$ (by (i))
$= 2^k\cdot2-2+1$
$= 2^{k+1}-1$

Therefore the given relation is true for $n=k+1$ when it is true for $n=k$,
Thus we can see that given relation $a_n=2^n-1$ is true for $n=2$ when it is true for $n=1$, Similarly since it is true for $n=2$ then it is true for $n=3$ and so on. Therefore it will be true for;
$n = 1, 2, 3, 4, \ldots\ldots$

Therefore it is proved that $a_n = 2^n-1$ for all $n \in \mathbb{N}$

7 Let if possible $L > M$. Then $L - M > 0$

we choose $\epsilon = L - M$, since $\lim_{n \to \infty} a_n = L$

So from the above $\epsilon > 0$ $\exists$ $K \in \mathbb{N}$ such that;

$$|x_n - L| < \epsilon \quad \forall n \geq K \text{ - - - - (1)}$$

From (i)

$$-\epsilon < x_n - L < \epsilon, \ \forall n \geq K$$

$$\Rightarrow L - \epsilon < x_n < L + \epsilon \quad \forall n \geq K.$$

$$\Rightarrow x_n > L - \epsilon \ \ \forall n \geq K \Rightarrow x_n > M, \ \ \forall n \geq K. \quad [L - \epsilon = M]$$

This is a contradiction, since $a_n \leq M$ $\forall n \geq K$.

So our assumption is wrong.

Hence $L \not> M$ ie. $L \leq M$

8. Suppose that $\lim_{n\to\infty} |a_n| = |L|$ for some ~~real~~ number L.

The above statement is not true.

let's example $a_n = (-1)^n$.

and $\lim_{n\to\infty} |a_n| = |(-1)^n| = |\pm 1| = 1$

but $\lim_{n\to\infty} a_n = +1$ and -1 $\Rightarrow$ This is ocillatory sequence
$= +1, -1$

Hence $\therefore$ Here we take $a_n = (-1)^n$

9. Given; f is differentiable at 'a' $\in \mathbb{R}$
$\Rightarrow$ f is continuous at 'a' (since differentiability Implies continuity)

$\Rightarrow$ If sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to 'a'
then $\{f(a_n)\}_{n\in\mathbb{N}}$ converges to $f(a)$
(Sequential criterion of continuity)

Since f is differentiable at a;
$\Rightarrow$ $f'(a)$ exist and $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$

consider the sequence. $\{a_n\}_{n\in\mathbb{N}}$
define the sequence.

$$b_n = \frac{f(a_n) - f(a)}{a_n - a}$$

then $\lim_{n\to\infty} b_n = f'(a)$

10. Let $a_n = \frac{n}{n+1}$

let there exist $m < 1$ such that a_n doesn't exceed m.

then $\exists$, $\epsilon > 0$.

s.t.

$$a_n < 1 - \epsilon \quad \forall n \in N$$

$$a_n - 1 < -\epsilon$$

$$\frac{n}{n+1} - 1 < -\epsilon$$

$$\frac{n-(n+1)}{n+1} < -\epsilon$$

$$\frac{n-n-1}{n+1} < -\epsilon$$

$$\frac{-1}{n+1} < \epsilon$$

$$\frac{1}{n+1} > \epsilon$$

$$\lim_{n\to\infty} \frac{1}{n+1} > \lim_{n\to\infty} \epsilon$$

$$0 > \epsilon$$

contradiction

hence $\lim_{n\to\infty} \frac{n}{n+1} = 1$